

Time Asymmetry and Chaos in General Relativity.

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Abstract

In this work the late-time evolution of Bianchi type *VIII* models is discussed. These cosmological models exhibit a chaotic behaviour towards the initial singularity and our investigations show that towards the future, far from the initial singularity, these models have a non-chaotic evolution, even in the case of vacuum and without inflation. These space-time solutions turn out to exhibit a particular time asymmetry. On the other hand, investigations of the late-time behaviour of type *VIII* models by another author have the result that chaos continues for ever in the far future and that these solutions have a time symmetric behaviour: this result was obtained using the approximation methods of Belinski, Khalatnikov and Lifshitz (*BKL*) and we try to find out a possible reason explaining why the different approaches lead to distinct outcomes. It will be shown that, at a heuristic level, the *BKL* method gives a valid approximation of the late-time evolution of type *VIII* models, agreeing with the result of our investigations.

1 Introduction.

The fundamental laws of physics are often said to be *time symmetric*: Newton's laws, Electromagnetism, Quantum Mechanics and Einstein's theory of gravity do not distinguish between past and future. However the phenomena taking place in our world display an *arrow of time*. It is of great interest to try to understand the physical origin of this *time asymmetry*. In this work we will focus on Einstein's theory of gravity and study how some solutions of the Einstein Field Equations (*EFE*) can distinguish between past and future. We will discuss the particular way in which the nature of the solutions at early times close to the initial singularity differs from the nature of the solution at late times, in the far future.

The solutions that will be discussed are the Bianchi type *VIII* models, which are among the most general homogeneous solutions to the *EFE*. The type *VIII* model has been studied mainly at early times, where the possibility of *chaotic* behaviour was investigated. This solution was found by *BKL* [1],[2],[3] and Misner C. [4] to exhibit chaotic mixmaster behaviour as one approaches the initial singularity. As clock time $t \rightarrow 0$ the type *VIII* model evolves through an infinity of Kasner (Bianchi type *I*) stages, in a chaotic way, see [5]. The question arises to see whether this chaotic mixmaster behaviour continues for ever in the far future.

P. Halpern [6] has used the methods of V. Belinski, I. M. Khalatnikov and I. Lifshitz (*BKL*) [1] to study the late-time behaviour of type *VIII* models and the result is that chaos continues for ever in the far future of these models. Our investigation in [7], [8] shows that this can not be the case. The chaotic mixmaster solution close to the initial singularity evolves into a *simpler non-chaotic solution*. This result is consistent with the recent analysis of Ringström H. in [9] and other investigations supporting this result are [10], [11], [12] and [13].

It is important to understand why the application of the *BKL* method in [6] yields a completely different result. Either there could be some problem with the application to the late-time regime, or either there could be something intrinsically wrong with the *BKL* method. *BKL* applied their method also to the study of the general inhomogeneous solution at early times close to the initial singularity, so it is of relevance to determine what the nature of the problem actually is

and to see whether the *BKL* approximation could be used in the late-time regime as well. In order to achieve this we will compare in some detail the analysis of P. Halpern in [6] with the investigation of [7], [8].

2 The time evolution of Bianchi type *VIII* models.

The Bianchi type *VIII* universe is one of the four most general models of the spatially homogeneous universes and it is an eternally expanding solution. In recent work it is shown that type *VIII* universes become the most general of the spatially homogeneous models when *space is compactified*, see [16]. One of the reasons to believe that spatial sections of cosmological models could be compact is that formulations of quantum cosmology require the volume of the spatial sections to be finite, in order for cosmological wave-functions to exist. So one could expect that it is most probable for a compact homogeneous universe to have type *VIII* symmetry.

In [6] Halpern P. argues that the chaotic behaviour characteristic of the early time evolution of type *VIII* models continues for ever in the far future and that in this case there would be no essential difference between past and future for these models. However our investigation in [7], [8] shows that the behaviour in the past is very different from that in the future.

Let us review how the Bianchi type *VIII* model evolves at late times, far from the initial singularity, according to [7], [8], [9], [10], [11], [12] and [13]. It turns out to be useful to discuss the problem in the *Hamiltonian formalism* [4]. In this formulation, using the so called metric approach [22], where the metric components are the basic variables of the gravitational field, one can introduce group-invariant and time-independent frame vectors e_a such that the line element for Bianchi class A models is given by, using an arbitrary time variable t'

$$ds^2 = -N(t')dt'^2 + g_{ab}W^aW^b$$

where W^a are time-independent one-forms dual to the frame vec-

tors e_a . We can introduce three time-dependent scale factors as

$$g_{ab} = \text{diag}(a^2, b^2, c^2)$$

which can be rewritten as

$$g_{ab} = \text{diag}(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}) \quad (1)$$

with

$$\begin{aligned} \beta_1 &= \beta^0 - 2\beta^+ \\ \beta_2 &= \beta^0 + \beta^+ + \sqrt{3}\beta^- \\ \beta_3 &= \beta^0 + \beta^+ - \sqrt{3}\beta^- \end{aligned}$$

The evolution of the type *VIII* model corresponds to the motion of a universe point in a triangular shaped potential in minisuperspace (β^+, β^-) , possessing an infinite open channel along the β^+ axis [22]. The motion of the universe point would then be analogous to motion of a ball in a triangular billiard, the difference being that the reflection angle before the bounce with a wall is not equal to the angle after the bounce. If the universe point moves initially in a straight line (corresponding to Kasner-like behaviour) it will bounce off a potential wall and then move along another straight line (a transition to another Kasner-like behaviour). The triangular shape of the potential allows the evolution to be *chaotic*. As the model evolves in the future, the Hamiltonian picture implies that the triangular potential contracts, thus enabling the universe point to bounce of the walls in a chaotic sequence, ad infinitum, such that mixmaster behaviour would continue for ever. Our analysis based on a combination of the Hamiltonian formalism and the orthonormal frame approach shows that this can not be the case. The universe point is forced to leave the triangular region of the potential and to escape along the infinite open channel along the β^+ axis, such that $\beta^- \rightarrow 0$ and $\beta^+ \rightarrow +\infty$. This late-time evolution will be characterised by an infinity of *non-chaotic oscillations* between the two walls of the channel. The open channel becomes increasingly narrow as $\beta^+ \rightarrow +\infty$ and the universe point will exhibit *increasingly rapid non-chaotic oscillations* about the axisymmetric type *VIII* solution. The line element for vacuum type *VIII* models will tend in this way to the Bianchi type *III* form of flat space-time,

$$ds^2 = -dt^2 + t^2(dx^2 + e^{2x}dy^2) + dz^2 \quad (2)$$

(the Bianchi type VI_h diagonal plane wave metric [22] with $h = -1$ reduces to this line element for a particular choice of parameters) such that the shear parameter Σ^2 ,

$$\Sigma^2 = \frac{\sigma^2}{3H^2}$$

(with σ^2 being the shear scalar and H being the Hubble parameter), exhibits *increasingly rapid non-chaotic oscillations*. For non-vacuum perfect fluid models, with equation of state $p = (\gamma - 1)\rho$ and $\gamma > 2/3$, the mixmaster behaviour will evolve as well to a *simpler non-chaotic solution* corresponding to the motion of the universe point along the infinite open channel of the potential. In this case the line element will tend to the Collins type *III* [22] solution if $2/3 < \gamma < 1$ and to the above vacuum type *III* line element if $1 \leq \gamma \leq 2$. In each of these cases the shear parameter Σ^2 exhibits *increasingly rapid non-chaotic oscillations*. Another remarkable property of these models is that the Weyl scalar [10], [12] is unbounded towards the far future: this might be of relevance in view of the relation between gravitational entropy and the Weyl curvature as conjectured by R. Penrose [19]. In this context, a further interesting study of the behaviour of the Weyl scalar for type *VIII* models and for other homogeneous solutions can be found in [20].

3 Is there a problem with the *BKL* approximation method ?

Let us study the analysis of Halpern P. [6] and try to understand why this work leads to different results.

First, in this work it is argued that in the Hamiltonian picture, the evolution of two neighbouring points in the type *VIII* potential is such that paths will diverge even if the points were close to each other initially. This is a feature of chaotic systems. Furthermore it is argued that this Hamiltonian system is *time reversible* and that therefore the

Bianchi type *VIII* model is chaotic both towards and away from the initial singularity. Our investigations show that the *time reversibility* can not be used to deduce the behaviour far from the singularity. The Bianchi type *VIII* system on the contrary exhibits *time asymmetry*, corresponding to the fact that as one approaches the singularity one has *chaotic mixmaster* behaviour while the far future is characterised by a simpler *non-chaotic* evolution.

Second, another argument giving different results in the analysis of [6] is the following. In [6] essentially the same method was used as in the work of *BKL* [1], giving the result that chaos continues for ever in the far future. So does this implies that there is a problem with the *BKL* method itself or has the method been applied incorrectly ? Or is the *BKL* approximation method not applicable in the late-time regime ?

In [6] the field equations

$$2(\ln a)'' = (b^2 + c^2)^2 - a^4 \quad (3)$$

$$2(\ln b)'' = (a^2 + c^2)^2 - b^4 \quad (4)$$

$$2(\ln c)'' = (a^2 - b^2)^2 - c^4 \quad (5)$$

$$(\ln a^2)'' + (\ln b^2)'' + (\ln c^2)'' = (\ln a^2)'(\ln b^2)' + (\ln b^2)'(\ln c^2)' + (\ln c^2)'(\ln a^2)' \quad (6)$$

(the prime denotes differentiation w.r.t. η defined by $dt = (abc)d\eta$) were studied with the assumption that one of the scale factors is smaller then the other two, in order to study the evolution of a Kasner stage. The transition from one Kasner epoch to the other was parametrised by writing Kasner exponents as functions of a single parameter u

$$p_1 = \frac{u}{1 - u + u^2}$$

$$p_2 = \frac{1 - u}{1 - u + u^2}$$

$$p_3 = \frac{u^2 - u}{1 - u + u^2}$$

with $u > 1$ and such that the scale factors correspond to

$$a \sim t^{p_1}$$

$$b \sim t^{p_2}$$

$$c \sim t^{p_3}$$

Supposing that initially $p_1 > p_3 > 0$ and $p_2 < 0$, and if one includes only the dominant terms in (3)-(6), we get that, as $t \rightarrow +\infty$, the field equations lead to another Kasner stage with $p'_3 > p'_2 > 0$ and $p'_1 < 0$ and so as the model evolves away from the singularity, the transition between Kasner epochs is given by the transformation

$$u \rightarrow u + 1 \tag{7}$$

Two of the scale factors will have exchanged their increasing and decreasing behaviour while the other scale factor continues to increase monotonically. Now, (7) means that the parameter u increases indefinitely, so it will never reach values $u < 1$, which implies that there is *no reason for a new Kasner era to begin* and the alternation between Kasner states seems to continue for ever. However, as the parameter u continues to increase, the Kasner parameters p_1 , p_2 and p_3 approach the following values

$$p_1 \sim \frac{1}{u}$$

$$p_2 \sim \frac{-1}{u}$$

$$p_3 \sim 1 - \frac{1}{u^2}$$

and thus the exponents p_1 and p_2 become close to each other. This means that the approximation done at the beginning of this calculation, namely keeping the dominant scale factor a only in the right-hand side of (3)-(5), is not valid any more since $a \sim b$. In [6] the following notation is used to write the resulting field equations : $\alpha = \ln a$, $\beta = \ln b$ and $\gamma = \ln c$. When taking both a and b into account, (3)-(5) can be written as

$$\alpha'' + \beta'' = 0$$

$$\alpha'' - \beta'' = e^{4\beta} - e^{4\alpha}$$

The equations (3)-(6) also yield a first integral which is given by, when terms containing c are neglected compared to those containing dominant contributions of a and b ,

$$\alpha'\beta' + \beta'\gamma' + \gamma'\alpha' = \frac{1}{4}(e^{2\alpha} + e^{2\beta})$$

The subsequent analysis in [6] follows the one of [1]: the following change of time variable is made

$$\xi = \xi_0 \exp \left[\frac{2a_0^2}{\xi_0^2} (\eta - \eta_0) \right] \quad (8)$$

with ξ_0 , a_0 and η_0 being constants. If we define further $\chi = \alpha - \beta$, then the above field equations can be written as [6], [1]

$$\chi_{\xi\xi} + \frac{1}{\xi}\chi_{\xi} + \frac{1}{2}\sinh 2\chi = 0 \quad (9)$$

$$\gamma_{\xi} = -\frac{1}{4\xi} + \frac{1}{8}\xi[\chi^2_{\xi} + \cosh 2\chi + 1] \quad (10)$$

Now it was argued that as $\eta \rightarrow +\infty$, we have $\xi \rightarrow +\infty$ because of equation (8), so that terms containing $\frac{1}{\xi}$ can be neglected in (9) and (10). If we assume that a and b are very close to each other, then we have $\sinh(2\chi) \approx 2\chi$ and these approximations lead to the following solution of (9)-(10)

$$a, b = a_0 \left(\frac{\xi}{\xi_0} \right)^{\frac{1}{2}} \exp \left[\pm \left(\frac{A}{\sqrt{\xi}} \right) \sin(\xi - \xi_0) \right] \quad (11)$$

$$c = c_0 \exp \left[\frac{1}{8}(\xi^2 - \xi_0^2) \right] \quad (12)$$

At some stage the increasing scale factor c will become comparable to a and b and will become the dominant term as $\xi \rightarrow +\infty$: this enables a *new Kasner era* to begin, with this time $c \gg a, b$. Further developments in [6] are done in order to show that this type of evolution is chaotic in the far future.

Now, the solution (11)-(12) is valid if $\xi \rightarrow +\infty$, which corresponds to $\eta \rightarrow +\infty$ because of equation (8). However as was explained in [7], a consequence of the inequality $(abc)'' > 0$ (which can be deduced

directly from the field equations (3)-(5), the dot denotes differentiation w.r.t. synchronous time t) can be that

$$\eta \rightarrow 0 \quad (13)$$

as $t \rightarrow +\infty$.

We can check this with the future asymptotic form of the line-element for vacuum Bianchi type *VIII* that we obtained, namely the Bianchi type *III* form of flat space-time given by (2). For this line element we have

$$abc \sim t^2$$

This means that the relation between the η -time and the synchronous time t is given by

$$\eta = \int \frac{dt}{abc} \sim \frac{-1}{t}$$

so that indeed $\eta \rightarrow 0$ as $t \rightarrow +\infty$.

This behaviour of *BKL* (η) time given by equation (13) has also been observed for type Bianchi *VII*₀ models in the work of Wainwright et al. [21].

This differs however with the derivation of *BKL* [1], (see their equation 4.18) where the integration of $dt = (abc)d\eta$ yielded

$$t \sim e^{\frac{1}{8}(\xi^2 - \xi_0^2)}$$

where ξ_0 is a constant, because this implies that $\xi \rightarrow +\infty$ as $t \rightarrow +\infty$. Such behaviour of ξ is also supposed to be true in the work of I. M. Khalatnikov and V. L. Pokrovsky [17]. This hypothesis is the source of the problem. Our result implies that $\xi \rightarrow \text{constant}$ as $t \rightarrow +\infty$, which means that far from the initial singularity the solution (11)-(12) is no longer valid, since $\frac{1}{\xi}$ -terms are not ignorable in equations (9)-(10). Indeed equation (9) with $\sinh(2\chi) \approx 2\chi$ is a Bessel equation, a solution being a Bessel function of order zero

$$\chi(\xi) = J_0(\xi)$$

The fact that ξ is assumed to continually increase is important in the analysis of [6] since this allows the scale factor c given by (12) to become larger than a and b , such that the intermediate axisymmetric

stage comes to an end in order for a new Kasner era to develop. As a result it was then explained in [6] that an "axisymmetric period" with two scale factors close to each other is finite in duration and that this ultimately has no effect on the solution.

However if $\xi \rightarrow \text{constant}$ as $t \rightarrow +\infty$, then $\chi(\xi) = J_0(\xi) \rightarrow \text{constant}$: the difference between α and β will not change. In addition, the solution of equation (10) for $\gamma(\xi)$ implies that $\gamma(\xi) \rightarrow \text{constant}$ as $\xi \rightarrow \text{constant}$, such that the scale factor c can not become greater than a and b as was supposed in [6]. In other words the axisymmetric regime could never come to an end, which agrees with the result of our investigation and with that of other authors.

Thus summarising one can say that provided one takes into account the correct interval for the *BKL* time variable η , i.e. $] -\infty, 0]$, then the *BKL* approximation method predicts that the vacuum Bianchi type *VIII* solution tends to an axisymmetric solution as $t \rightarrow +\infty$. Thus as heuristic or approximative analysis the *BKL* method does not fail in the late-time regime.

The rigorous analysis of Ringström H. in [18] of the Bianchi type *IX* model close to the singularity gave support to the validity of the *BKL* approximation method. In fact we have a new support to the heuristic validity of the *BKL* approximation method at late times, since it leads basically to a similar result as other rigorous methods [9], [10] and [12].

4 Chaos disappears in the far future.

The fact that chaotic behaviour disappears at late times such that the type *VIII* models evolve into a simpler solution even when $\gamma > 2/3$, thus *without inflation*, is remarkable in the sense that for most complex physical systems chaos is a feature of *the whole evolution*, for instance in the case of the three-body problem there seems to be no time direction in which chaos ceases to exist. In [15] it is explained that there is a frequently overlooked distinction between irreversible and chaotic behaviour of Hamiltonian systems. The latter does not in general appears to possess a direction of time, i.e. there is no essential distinction between past and future. By looking at a sequence of "snapshots" or configurations of the system at different instants

of time, one would not observe some asymmetry. The general solutions to the Einstein Field Equations that we discussed do not exhibit this distinction between chaos and time asymmetry, since the time evolution of the type *VIII* models distinguishes between past and future. The answer to the question "Is the type *VIII* system chaotic?" should then be: "It depends on the direction of time: towards the initial singularity the evolution is chaotic, while in the far future it is non-chaotic". The type *VIII* system appears to loose memory of the chaotic initial state, and evolves in the far future to the axisymmetric type *VIII* solution. Thus the simple late-time asymptotic solution can be seen as arising from a wider set of (chaotic) initial conditions. This time asymmetry corresponds to the one defined by Halliwell J. in [14], which is based on the distinction between the metric at early and late times. An important point to note is that the singularity of the type *VIII* solutions plays a crucial role in the dynamics of these models : some terms in the field equations become always negligible (such that Kasner stages are always possible) close to the singularity, while far from the singularity they are not (see previous section). This is related to the time asymmetry exhibited by the type *VIII* model.

Finally, let us point out that the asymmetry between past and future of type *VIII* models might be of relevance in the study of the late-time behaviour of cosmological solutions of (super)string theory as studied in [23], where mixmaster behaviour was found to occur close to the initial singularity.

5 Conclusion.

In the work of *BKL* [1] and Halpern [6] the interval of variation of the *BKL* time variable η is supposed to be $] - \infty, +\infty[$. As a consequence the *BKL* approximation method predicts mixmaster behaviour to occur in vacuum type *VIII* models at late times, far from the initial singularity. In this case the whole evolution of the type *VIII* model would be chaotic. If one takes into account that the correct interval of variation for η is $] - \infty, 0[$ then the *BKL* method predicts that at late times the type *VIII* model tends to the axisymmetric solution, thus in agreement at a heuristic level with other rigorous meth-

ods. The *BKL* method is thus a valid approximation at late times and it could be applied in order to obtain information about the nature of the general vacuum inhomogeneous solution in the late-time regime. The late-time evolution of type *VIII* models is non-chaotic, thus they distinguish between past and future. Remarkably the type *VIII* model loses memory of the chaotic initial state and evolves to a simpler non-chaotic future state. One could expect or conjecture that a similar time asymmetry will be found in the evolution of the general vacuum inhomogeneous solution. This remarkable type of dynamical feature of the Einstein Field Equations is not present in Newtonian or non-relativistic physics and is closely related to the occurrence of singularities in these space-time solutions.

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